COMMENTS ON "INTRODUCTION TO P-ADIC TEICHMÜLLER THEORY"

Shinichi Mochizuki

March 2022

(1.) In the second paragraph of §2.2, the phrase "generalize the the" should read "generalize the".

(2.) In the second paragraph of §2.5, the notation "$\mathrm{End}_{\mathcal{O}_\lambda}(G_\lambda)$" should read "$\mathrm{End}_{\mathcal{O}_\lambda}(\mathcal{G}_\lambda)$".

(3.) With regard to the notation "$\mathcal{N}^{\log} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$", "$\mathcal{C}^{\log} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$" in the paragraph immediately preceding Theorem 1.4, we note the following: Let K be a finite extension of $\mathbb{Q}_p$ and $\mathfrak{Y}$ a formally smooth p-adic formal scheme over the ring of integers $\mathcal{O}_K$ of K, i.e., such as a suitable étale localization of $\mathcal{N}$ or $\mathcal{C}$. Then $\mathfrak{Y} \times_{\mathbb{Z}_p} \mathbb{Q}_p$ (i.e., "$\mathfrak{Y} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$") is defined as the *ringed space* obtained by tensoring the structure sheaf of $\mathfrak{Y}$ over $\mathcal{O}_K$ with K. Thus, if, for instance, $\mathfrak{Y}$ is the formal scheme obtained as the formal inverse limit of an inverse system of schemes

$$\ldots \hookrightarrow \mathfrak{Y}_n \hookrightarrow \mathfrak{Y}_{n+1} \hookrightarrow \ldots$$

— where n ranges over the positive integers, and each "$\hookrightarrow$" is a nilpotent thickening — and U is an affine open of the *common* underlying topological space of the $\mathfrak{Y}_n$, then the rings of sections of the respective structure sheaves $\mathcal{O}_{\mathfrak{Y}}$, $\mathcal{O}_Y$ of $\mathfrak{Y}$, Y over U are, by definition, given as follows:

$$\mathcal{O}_{\mathfrak{Y}}(U) \stackrel{\mathrm{def}}{=} \varprojlim_n \mathcal{O}_{\mathfrak{Y}_n}(U); \quad \mathcal{O}_Y(U) \stackrel{\mathrm{def}}{=} \mathcal{O}_{\mathfrak{Y}}(U) \otimes_{\mathcal{O}_K} K.$$

Here, we observe that $\mathcal{O}_{\mathfrak{Y}}(U)$ is the *p-adic completion* of a *normal noetherian ring of finite type over* $\mathcal{O}_K$. In particular, we observe that one may consider *finite étale coverings* of Y, i.e., by considering *systems of finite étale algebras* $\mathcal{A}_U$ over the various $\mathcal{O}_Y(U)$ [that is to say, as U is allowed to vary over the affine opens of the $\mathfrak{Y}_n$] equipped with *gluings* over the intersections of the various U that appear. Note, moreover, that by considering the *normalizations* of the $\mathcal{O}_{\mathfrak{Y}}(U)$ in $\mathcal{A}_U$, we conclude [cf. the discussion of the Remark immediately following Theorem 2.6 in Section II of [1]] that

(NorFor) any such system $\{\mathcal{A}_U\}_U$ may be obtained as the $W \stackrel{\text{def}}{=} \mathfrak{W} \times_{\mathcal{O}_K} K$ for some *formal scheme* $\mathfrak{W}$ that is *finite* over $\mathcal{Y}$, and that arises as the *formal inverse limit* of an inverse system of schemes

$$\ldots \hookrightarrow \mathfrak{W}_n \hookrightarrow \mathfrak{W}_{n+1} \hookrightarrow \ldots$$

— where n ranges over the positive integers; each "$\hookrightarrow$" is a nilpotent thickening; for each affine open V of the *common* underlying topological space of the $\mathfrak{W}_n$, $\mathcal{O}_{\mathfrak{W}}(V)$ is the *p-adic completion* of a *normal noetherian ring of finite type over* $\mathcal{O}_K$.

Indeed, this follows from well-known considerations in commutative algebra, which we review as follows. Let R be a *normal noetherian ring of finite type over a complete discrete valuation ring* A [i.e., such as $\mathcal{O}_K$ in the above discussion] with *maximal ideal* $\mathfrak{m}_A$ and *quotient field* F such that R is *separated* in the $\mathfrak{m}_A$-adic topology. Thus, since A is *excellent* [cf. [2], Scholie 7.8.3, (iii)], it follows [cf. [2], Scholie 7.8.3, (ii)] that R is *excellent*, hence that the $\mathfrak{m}_A$-adic completion $\widehat{R}$ of R is also *normal* [cf. [2], Scholie 7.8.3, (v)]. Then it is well-known and easily verified [by applying a well-known argument involving the *trace map*] that the *normalization* of $\widehat{R}$ in any *finite étale algebra* over $\widehat{R} \otimes_A F$ is a *finite algebra* over $\widehat{R}$. Let $\widehat{S}$ be such a *finite algebra* over $\widehat{R}$. Then it follows immediately from a suitable version of *"Hensel's Lemma"* [cf., e.g., the argument of [3], Lemma 2.1] that $\widehat{S}$ may be obtained, as the notation suggests, as the $\mathfrak{m}_A$-adic completion of a *finite algebra* S over R, which may in fact be assumed to be *separated* in the $\mathfrak{m}_A$-adic topology and [by replacing S by its normalization and applying [2], Scholie 7.8.3, (v), (vi)] *normal*. Let $f \in R$ be an element that maps to a *non-nilpotent* element of $R/\mathfrak{m}_A \cdot R$. Write $R_f \stackrel{\text{def}}{=} R[f^{-1}]$; $S_f \stackrel{\text{def}}{=} S \otimes_R R_f$; $\widehat{R}_f$, $\widehat{S}_f$ for the respective $\mathfrak{m}_A$-adic completions of R_f, S_f. Then it follows again from [2], Scholie 7.8.3, (v), that $\widehat{S}_f$, which may be naturally identified [since S is a *finite algebra* over R] with $\widehat{S} \otimes_{\widehat{R}} \widehat{R}_f$, is *normal*. That is to say, it follows immediately that

(NorForZar) the operation of forming *normalizations* [i.e., as in the above discussion] is *compatible* with *Zariski localization* on the *given formal scheme*.

On the other hand, one verifies immediately that (NorFor) follows formally from (NorForZar).

Bibliography

[1] G. Faltings, Crystalline Cohomology and p-adic Galois Representations, *Proceedings of the First JAMI Conference*, Johns Hopkins Univ. Press (1990), pp. 25-79.

[2] A. Grothendieck and J. Dieudonné, Éléments de géométrie algébrique IV, Étude locale des schémas et des morphismes de schémas, Séconde partie, *Publ. Math. IHES* **24** (1965).

[3] S. Mochizuki, Topics in Absolute Anabelian Geometry II: Decomposition Groups and Endomorphisms, *J. Math. Sci. Univ. Tokyo* **20** (2013), pp. 171-269.